MAGNETOACOUSTIC WAVES IN HETEROGENEOUS MEDIA

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We consider the propagation of magnetoacoustic waves in an ideally conductive laminar-heterogeneous medium in the presence of a constant external magnetic field. The equation for determination of the reflection factor of a fast magnetoacoustic wave from an heterogeneous layer is obtained.

1. Let the properties of the medium continuously change in the direction of z-axis while the external magnetic field H is perpendicular to this axis. Let us direct the x-axis along the vector H. The linearized equations of the magnetohydrodynamics for the waves, polarized in the plane xz, take the form [1]

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\partial h_x}{\partial t} = -H\frac{\partial v_z}{\partial z}, \qquad \frac{\partial h_z}{\partial t} = H\frac{\partial v_z}{\partial x}$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{H}{4\pi\rho}\left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right)$$

$$\frac{\partial p}{\partial t} + \rho a^2\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right) + a^2 v_z \frac{\partial \rho}{\partial z} = 0 \tag{1.1}$$

Here v is the velocity of the medium, p and h are, respectively, small changes in pressure and in the magnetic field in the wave. The density ρ and the normal velocity of sound a, defining properties of the medium, are functions of the coordinate z. By virtue of adiabatic nature of the motion, changes in pressure and in density are connected by the relation

$$\frac{\partial p}{\partial t} = a^2 \frac{\partial \rho}{\partial t}$$

We shall assume that the unperturbed parameters of the medium vary in space sufficiently slowly. Therefore, the last term of the fifth equation in (1.1) being a small value of second order can be neglected.

In the case of monochromatic waves of frequency ω, eliminating v_x, h_x and h_z from (1.1), we have

$$\frac{\partial}{\partial z}\left[(1+\psi)\,p + \psi\frac{a^2}{\omega^2}\frac{\partial^2 p}{\partial x^2}\right] = i\omega\rho\left(v_z + \psi\frac{a^2}{\omega^2}\frac{\partial^2 v_z}{\partial x^2}\right)$$

$$\frac{\partial v_z}{\partial z} = \frac{i\omega}{\rho a^2}\left(p + \frac{a^2}{\omega^2}\frac{\partial^2 p}{\partial x^2}\right), \qquad \psi = \frac{H^2}{4\pi\rho a^2} \tag{1.2}$$

Assuming

$$p = P(z)\, e^{i(bx-\omega t)}, \qquad v_z = V(z)\, e^{i(bx-\omega t)}, \qquad b = \text{const} \tag{1.3}$$

system (1.2) is reduced to

$$\frac{\partial}{\partial z}(up) = i\omega g v_z, \qquad \frac{\partial v_z}{\partial z} = \frac{i\omega}{\rho a^2}(1-\chi)\,p \tag{1.4}$$

Here the following notation is introduced

$$\chi = (ab/\omega)^2, \qquad g = \rho(1-\psi\chi), \qquad u = 1 + \psi - \psi\chi \tag{1.5}$$

From (1.4) taking into account (1.3) we obtain the following equation for pressure:

$$\Delta p + \left(k^2 + \frac{u''}{u'} - \frac{g'}{g}\frac{u'}{u}\right)p + \left(2\frac{u'}{u} - \frac{g'}{g}\right)p' = 0 \tag{1.6}$$

Here derivatives with respect to z are denoted by primes, Δ is the Laplace operator,

$$k^2 = \frac{\omega^2}{a^2 u} \tag{1.7}$$

At a very slow change of unperturbed parameters of the medium in space, when terms with the derivatives of these parameters may be disregarded, Eq. (1.6) takes the form of a wave equation with the variable wave number k

$$\Delta p + k^2 p = 0 \tag{1.8}$$

Quantity u plays part of a square of dimensionless velocity of the magnetoacoustic wave in a heterogeneous medium. According to (1.5) and (1.7) we have

$$u^2 - (1 + \psi)\, u + \psi\, (b/k)^2 = 0 \tag{1.9}$$

Let us assume that when $z \to -\infty$, parameters of the medium tend to constant values ρ_0 and a_0. Dispersion equation of plane magnetoacoustic waves in a homogeneous medium has the form [1] $u_0^2 - (1 + \psi_0)\, u_0 + \psi_0 \sin^2 \theta_0 = 0 \quad \left(u_0 = \frac{\omega^2}{a_0^2 k_0^2}, \quad \psi_0 = \frac{H^2}{4\pi\rho_0 a_0^2}\right)$

where θ_0 is the angle between the normal to the wave front and the oz-axis. (1.10)

Since for $z \to -\infty$ the quantity u must tend to u_0, from (1.9) and (1.10) follows

$$b = k_0 \sin \theta_0 \tag{1.11}$$

According to the Snell law

$$k_0 \sin \theta_0 = k \sin \theta \tag{1.12}$$

and Eq. (1.9) takes the form

$$u^2 - (1 + \psi)\, u + \psi \sin^2 \theta = 0 \tag{1.13}$$

The roots of this equation determine velocities of fast and slow magnetoacoustic waves in a heterogeneous medium.

Since discarding of small terms in a differential equation can, in some cases, lead to erroneous results, the case presents an interest to reduce the exact equation (1.6) to a wave equation.

This is possible if in place of pressure, a new function is introduced

$$T = \frac{pu}{\sqrt{g}} \tag{1.14}$$

For this function, in accordance with (1.6), the following wave equation with a certain "effective" square of the wave number is obtained:

$$\Delta T + k_{э\Phi}^2 T = 0$$
$$k_{э\Phi}^2 = k^2 + \frac{g''}{2g} - \frac{3}{4}\left(\frac{g'}{g}\right)^2 \tag{1.15}$$

For $\psi = 0$

$$T = \frac{p}{\sqrt{\rho}}, \quad k_{э\Phi}^2 = k^2 + \frac{\rho''}{2\rho} - \frac{3}{4}\left(\frac{\rho'}{\rho}\right)^2, \quad k^2 = \frac{\omega^2}{a^2} \tag{1.16}$$

which coincides with the known result for acoustic waves in heterogeneous media in the absence of magnetic field [2, 3].

2. Now let parameters of the medium also tend to constant values ρ_1, a_1 for $z \to +\infty$, i.e. we have a heterogeneous layer. Let us assume that from a homogeneous medium with $z = -\infty$ a plane fast magnetoacoustic wave is propagating in the direction of positive z.

Equations (1.4) can be satisfied if we suppose thay for $z = -\infty$, besides an incident wave there exists a reflected wave as well.

Our problem consists in determining the relation of amplitudes of reflected and incident waves, i.e. to find the amplitude reflection factor. In the case under consideration of a magnetic field perpendicular to the z-axis, it is possible to obtain a special equation for this factor (similar to the case of acoustic waves in the absence of a magnetic field and the case of electromagnetic waves [3]).

Setting in (1.4)

$$p = Z(z)v_z \tag{2.1}$$

and eliminating the derivative p', we obtain the Rikkati equation for determination of function $Z(z)$

$$Z' + \frac{u'}{u}Z + \frac{i\omega}{\rho a^2}(1-\chi)Z^2 = \frac{i\omega g}{u} \tag{2.2}$$

If the medium properties vary slightly on the extent of the wave length, the first two terms in the left side of (2.2) can be neglected, and we then find

$$Z = \pm\left(\frac{\rho a^2 g}{u(1-\chi)}\right)^{1/2} = \pm\frac{\rho a}{\sqrt{u}}\left(\frac{u-\psi\sin^2\theta}{u-\sin\theta}\right)^{1/2} \tag{2.3}$$

The "plus" sign corresponds to an incident wave and the "minus" sign to a reflected wave.

The more accurate value of the function $Z(z)$ differs from (2.3) by small terms of the order of derivatives of unperturbed parameters of the medium. These additional terms must be discarded as with substitution of $Z(z)$ in (2.1) they will give quantities of the second order of smallness.

Let us now define incident and reflected waves in the following manner:

incident wave

$$p = P(z)e^{i(bx-\omega t)}, \qquad v_z = Z^{-1}P(z)e^{i(bx-\omega t)}$$

reflected wave

$$p = R(z)e^{i(bx-\omega t)}, \qquad v_z = -Z^{-1}R(z)e^{i(bx-\omega t)}$$

Here, by Z we understand its value with the "plus" sign.

According to (1.4) with (2.3) taken into account, we have for functions P and R the following equations:

$$P' - \left[\frac{i\omega g}{Zu} + \frac{1}{2}\left(\frac{Z'}{Z} - \frac{u'}{u}\right)\right]P + \frac{1}{2}\,\frac{(Zu)'}{Zu}R = 0$$

$$R' + \frac{1}{2}\frac{(Zu)'}{Zu}P + \left[\frac{i\omega g}{Zu} - \frac{1}{2}\left(\frac{Z'}{Z} - \frac{u'}{u}\right)\right]R = 0$$

Multiplying the first of the above equations by R, and the second by P, subtracting one from the other and dividing the result by P^2, we obtain the Rikkati equation for the reflection factor $W(z) = R/P$

$$W' = -2i\beta W + \gamma(1 - W^2) \tag{2.4}$$

$$\beta = \frac{\omega g}{Zu} = k_0\sqrt{n^2 - \sin^2\theta_0}, \qquad \gamma = -\frac{1}{2}\,\frac{(Zu)'}{Zu} = \frac{1}{2}\,\frac{g}{\beta}\left(\frac{\beta}{g}\right)' \tag{2.5}$$

$$n = \frac{k}{k_0} = \frac{a_0\sqrt{u_0}}{a\sqrt{u}}$$

As the boundary condition we have

$$W \to 0 \quad \text{for} \quad z \to \infty \tag{2.6}$$

since for $z \to \infty$ (behind the layer) the reflected wave is absent.

Equation (2.4) differs from the equation for the reflection factor of the acoustic wave in the absence of magnetic field only by the fact that in place of density ρ the quantity g appears, and the velocity of the magnetoacoustic wave replaces the normal velocity of sound.

In a weak magnetic field ($\psi_0 \ll 1$) according to (1.5), (1.10), (1.13) and (2.5) we have

$$u_0 = 1 + \psi_0 \cos^2 \theta_0, \qquad u = 1 + \psi_0 (\rho_0/\rho) \ (n^2 - \sin^2 \theta_0)$$
$$\beta = \beta_{(0)} (1 + \psi_0 \delta), \qquad \gamma = \gamma_{(0)} + {}^1/_2 \psi_0 [\delta + (\rho_0/\rho) \sin^2 \theta_0]' \tag{2.7}$$
$$\beta_{(0)} = \frac{\omega}{a_0} \sqrt{n^2 - \sin^2 \theta_0}, \qquad \gamma_{(0)} = \frac{1}{2} \frac{\rho}{\beta_{(0)}} \left(\frac{\beta_{(0)}}{\rho}\right)'$$
$$\delta = \frac{1}{2} \left(\frac{\sin^2 \theta_0 \cos^2 \theta_0}{n^2 - \sin^2 \theta_0} - n^2 \frac{\rho_0}{\rho}\right), \qquad n = \frac{a_0}{a}$$

Setting in (2.4)

$$W = W_{(0)} + \psi_0 \varepsilon (z) \tag{2.8}$$

where $W_{(0)}$ is the reflection factor in the absence of the magnetic field, and discarding terms of the order of ψ_0^2, we obtain the following linear equation for the function $\varepsilon (z)$:

$$\varepsilon' + N\varepsilon = Q, \qquad N = 2 \ (i\beta_{(0)} + \gamma_{(0)} W_{(0)})$$
$$Q = - 2i\beta_{(0)}\delta W_{(0)} + {}^1/_2 \ [\delta + (\rho_0/\rho) \sin^2 \theta_0]' \ (1 - W_{(0)}^2)$$

Hence with (2.6) taken into account, we have

$$\varepsilon (z) = e^{-s} \int_{\infty}^{z} Q e^{s} dz, \qquad s = \int_{z_0}^{z} N dz \tag{2.9}$$

Here z_0 is an arbitrary fixed value of the coordinate z.

In a strong magnetic field ($\psi_0 \gg 1$)

$$u_0 = \psi_0, \qquad u = \psi = \psi_0 \frac{\rho_0 a_0^2}{\rho a^2} \qquad \beta = \frac{\omega}{a_0 \sqrt{\psi_0}} \sqrt{n^2 - \sin^2 \theta_0} \tag{2.10}$$
$$\gamma = - \frac{\beta'}{2\beta} = - \frac{nn'}{2 (n^2 - \sin^2 \theta_0)}, \qquad n^2 = \frac{\rho}{\rho_0}$$

If ψ_0 is so great that in (2.4) the term with the factor β, can be neglected, we obtain an equation with separable variables. By integrating this equation under condition (2.6) we find

$$W \equiv W_{\infty} = \frac{\sqrt{\rho_1} \cos \theta_1 - \sqrt{\rho} \cos \theta}{\sqrt{\rho_1} \cos \theta_1 + \sqrt{\rho} \cos \theta} \tag{2.11}$$

Expression (2.11) represents the reflection factor from the interface of two media [4], i.e. corresponds to the case of transitional layer of thickness considerably less than the length of a fast magnetoacoustic wave. Setting now in (2.4)

$$W = W_{\infty} + \frac{1}{\sqrt{\psi_0}} \eta(z) \tag{2.12}$$

and discarding terms of the order of ψ_0^{-1} we have

$$\eta' + 2\gamma W_{\infty} \eta = - 2i (\omega/a_0) W_{\infty} \sqrt{n^2 - \sin^2 \theta_0}$$

Hence

$$\eta = - 2i \frac{\omega}{a_0} e^{-s} \int_{\infty}^{z} W_8 \ \sqrt{n^2 - \sin^2 \theta_0} \, e^{s} dz, \qquad s = 2 \int_{z_0}^{z} \gamma W_{\infty} dz \tag{2.13}$$

In the case being considered of the incidence of a fast magnetoacoustic wave on an heterogeneous layer in a magnetic field, parallel to the layer, slow magnetoacoustic waves do not appear. In accordance with [4, 5], in such a magnetic field slow waves do not appear even in the presence of boundaries on which the medium properties change with a jump.

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ON EQUATIONS OF THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER OF BODIES OF REVOLUTION

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The uniformly accurate equations of a plane uniform laminar boundary layer for a body whose profile is sharply curved, are derived in [1]. In the present paper the results of [1] are generalized for the case of a body of revolution in a supersonic gas flow at incidence. The most important result lies in the fact that parameters of the gas flow in the boundary layer in the domain of sharp curvature of the generatirix of the body of revolution can be defined independently in every meridional plane passing through the axis of symmetry of the body if the curvature radius of the generatrix of the body becomes a quantity of the order of boundary-layer thickness.

1. We consider a certain body of revolution whose curvature $\varkappa$ of the

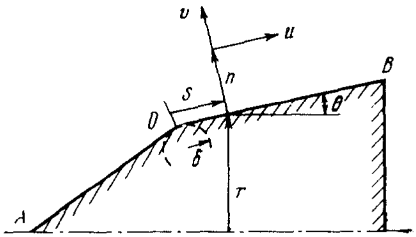

Fig. 1

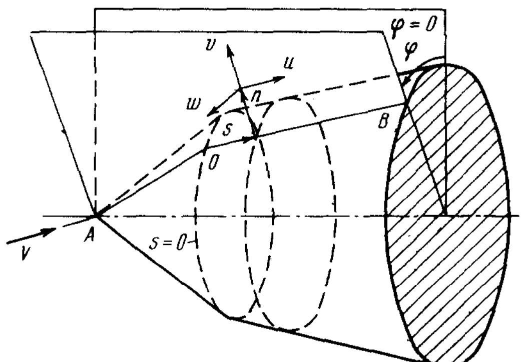

Fig. 2

generatrix AOB (Fig. 1) is a continuous function of the coordinate s, measured along the generatrix from the point O, where $\varkappa$ attains its greatest value $\varkappa_{max}$, and the radius of curvature, correspondingly, its minimum value $\delta = (\varkappa_{max})^{-1}$. We take the distance